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LETTER TO THE EDITOR

Glauber dynamics of neural network models

H Rieger, M Schreckenberg and J Zittartz

Institut für Theoretische Physik, Universität zu Köln, D-5000 Köln, Federal Republic of Germany

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Abstract. The Glauber dynamics of the Little-Hopfield model and the asymmetric SK model is studied with the help of path integrals introduced recently by Sommers. This dynamical approach confirms the results of Amit *et al.* For the fully asymmetric SK model one can calculate the response and autocorrelation functions exactly and observe exponential decay for both functions. The effect of weak correlations between the couplings J_{ij} and J_{ji} is investigated.

The connection of neural network models of the Little-Hopfield type [1] and Sherrington-Kirkpatrick spin glasses has been an active research field [2]. Recently, Sommers [3] introduced a path integral approach to Ising spin glasses. This letter tries to apply the Sommers method to the Little-Hopfield model and to find out if the results agree with those derived earlier [4] by the replica trick. Moreover, we discuss the implications for a model intermediate between symmetric and fully asymmetric synaptic couplings, similar to that of Hertz *et al* [5].

The advantage of a dynamical approach to the Little-Hopfield model is that it is possible to avoid the unphysical replica trick $n \to 0$ which is necessary in the static approach to perform the quenched average over the $p = \alpha N$ random patterns $\{\xi^\nu\}^{\nu=1,\dots,p}$. We will examine the relationship between time-persistent quantities and static order parameters introduced by Amit *et al* [4]. We write the spin distribution as a functional integral [3] and perform the quenched average over the $p-s$ non-condensed patterns [4] which leads to the introduction of the averaged spin autocorrelation function

$$C(t_1, t_2) = \frac{1}{N} \sum_{i=1}^{N} \overline{\langle \sigma_i(t_1)\sigma(t_2)\rangle} \tag{1}$$

the response function

$$G(t_1, t_2) = \frac{1}{N} \sum_{i=1}^{N} \overline{\frac{\delta}{\delta h_i(t_2)}} \langle \sigma_i(t_1)\rangle \tag{2}$$

the random-overlap correlation function

$$R(t_1, t_2) = \frac{1}{\alpha} \sum_{\mu=s+1}^{\alpha N} \overline{\langle m^\mu(t_1) m^\mu(t_2)\rangle} \tag{3}$$

and a function $S(t_1, t_2)$ related to $R(t_1, t_2)$ in the same way as $G(t_1, t_2)$ to $C(t_1, t_2)$ which is described below.

$$m^\mu(t) = \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \sigma_i(t) \qquad \mu = s+1, \ldots, p$$

is the random overlap of the network with one of the $p-s$ non-condensed patterns at time t. The bar means the average over the quenched disorder. Following Sommers [3] we choose homogeneous initial conditions for the spin distribution, let the initial time go to $-\infty$ and arrive at the generating functional for the spin correlation and response functions [6]:

$$\mathscr{Z}[\mathrm{i}\hat{\sigma}, b] = \left\langle\!\!\left\langle \left[\exp\left(\frac{\alpha}{2} \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, \tilde{R}(\tau_1 - \tau_2) \frac{\delta}{\delta h(\tau_1)} \frac{\delta}{\delta h(\tau_2)} + \alpha \int_{\tau_1 > \tau_2} \mathrm{d}\tau_1\, \mathrm{d}\tau_2\, S(\tau_1 - \tau_2) \frac{\delta}{\delta h(\tau_1)} \frac{\delta}{\delta \mathrm{i}\hat{\sigma}(\tau_2)} \right) \times \exp\left(\int \mathrm{d}\tau\, \mathrm{i}\hat{\sigma}(\tau) m(\tau) \right) \right] \right|_{h(\tau) = b + z\sqrt{\alpha r} + \sum_{\nu=1}^{s} m^\nu \xi^\nu} \right\rangle\!\!\right\rangle \tag{4}$$

where $\langle\!\langle \ldots \rangle\!\rangle$ means the average with respect to the s condensed patterns $\xi^1, \ldots, \xi^s$ and with respect to the Gaussian variable z with zero mean and variance one. $\tilde{R}(t)$ and r are the short-time and time-persistent parts of $R(t)$ respectively, which are defined below. $m(t)$ obeys the integral equation

$$m(t) = \int_{-\infty}^{t} \mathrm{d}\tau\, \mathrm{e}^{-\Gamma(t-\tau)} [\Gamma \tanh \beta h(\tau) + \mathrm{i}\hat{\sigma}(\tau)(1 - m^2(\tau))] \tag{5}$$

and the other quantities have to be determined self-consistently in a way described below (Γ is the spin-flip rate, β the inverse temperature and b a homogeneous external field). The self-consistence equations are given by

$$\begin{aligned} C(t_1 - t_2) &= \frac{\delta}{\delta \mathrm{i}\hat{\sigma}(t_1)} \frac{\delta}{\delta \mathrm{i}\hat{\sigma}(t_2)} \mathscr{Z}[\mathrm{i}\hat{\sigma}, b] \bigg|_{\hat{\sigma}=0} \\ G(t_1 - t_2) &= \frac{\delta}{\delta \mathrm{i}\hat{\sigma}(t_1)} \frac{\delta}{\delta h(t_2)} \mathscr{Z}[\mathrm{i}\hat{\sigma}, b] \bigg|_{\hat{\sigma}=0} \end{aligned} \tag{6}$$

and the relation between these functions and the Fourier transforms of the functions $\tilde{R}(t)$ and $S(t)$ in the exponent of the RHS of equation (4) is determined from the saddle point equations

$$\begin{aligned} S(\omega) &= \frac{1}{1 - G(\omega)} \\ R(\omega) &= \tilde{R}(\omega) + r \cdot \delta(\omega) = \frac{C(\omega)}{|1 - G(\omega)|^2} \end{aligned} \tag{7}$$

where we have defined the time-persistent part of $R(t)$: $r = \lim_{t\to\infty} R(t)$, and the short-time part of $R(t)$: $\tilde{R}(t) = R(t) - r$. It follows

$$r = \frac{q}{(1 - G(\omega = 0))^2} \tag{8}$$

where $q = \lim_{t\to\infty} C(t)$ is the EA order parameter. One can prove the existence of a fluctuation-dissipation theorem (FDT) between the short-time part $\tilde{C}(t)$ of the correlation function, i.e. $\tilde{C}(t) = C(t) - q$, and the response function $G(t)$ [3]. Then from equation (7) follows an FDT between the short-time part $\tilde{R}(t)$ of $R(t)$ and the random-overlap response $S(t)$:

$$\tilde{R}(\omega) = \frac{2}{\beta\omega} \operatorname{Im} S(\omega). \tag{9}$$

This enables us to extract the static properties in the same way as in the SK model [3]. We get the same equations for the order parameters q, r and m^ν ($\nu = 1, \ldots, s$) as Amit *et al* [4] within their replica symmetric solutions. It is important to observe that the local field $h(t)$ consists of four parts: an external field $b(t) = b$, a field $\Sigma_{\nu=1}^{s} m^\nu(t)\xi^\nu$ caused by the s macroscopic overlaps of the network with the learned patterns, a fluctuating part $\sqrt{\alpha}\Phi(t)$ with Gaussian correlations $\langle\Phi(t)\rangle = 0$ and $\langle\Phi(t)\Phi(t')\rangle = R(t-t')$ and not—as in the SK model—with $\langle\Phi(t)\Phi(t')\rangle = C(t-t')$, and a response part $\int_{\tau_1>\tau_2} \alpha S(\tau_1 - \tau_2)\sigma(\tau_2)$, where the random-overlap response functions $S(t-t')$—not the response function $G(t-t')$ as in the SK model—couples the neuron's value at time t to its values at former times $t' < t$. That is the reason why the situation in the Little–Hopfield model is a little more complicated than in the SK model. The first-order perturbation theory approximation for the low-frequency behaviour of the response function diverges for $\omega \neq 0$ near the generalised Almeida–Thouless line found by Amit *et al* [4]. We conclude that the above solution of the saddle-point equations in the static limit corresponds to the replica symmetric solution, which becomes unstable below the generalised AT line. This situation is quite similar to that of the SK model of spin glasses treated by Sommers [3].

In a further investigation we have studied the Glauber dynamics of the asymmetric SK model [5]. In this neural network model pairs of synaptic couplings (J_{ij}, J_{ji}) are independent and randomly distributed according to a bivariate Gaussian distribution with

$$\begin{aligned} \langle J_{ij}\rangle &= \langle J_{ji}\rangle = 0 \\ \langle J_{ij}^2\rangle &= \langle J_{ji}^2\rangle = J^2 \\ \langle J_{ij}J_{ji}\rangle &= J^2\lambda. \end{aligned} \tag{10}$$

The correlation parameter varies between -1 and $+1$. The case $\lambda = 1$ is the conventional Sherrington–Kirkpatrick model for spin glasses; the case $\lambda = 0$ represents the fully asymmetric SK model. The generating functional is now given by [6]:

$$\mathscr{Z}[\mathrm{i}\hat{\sigma}, b] = \left[\exp\left(\frac{J^2}{2}\int_{\tau_1,\tau_2} C(\tau_1-\tau_2)\frac{\delta}{\delta h(\tau_1)}\frac{\delta}{\delta h(\tau_2)} + \lambda J^2 \int_{\tau_1,\tau_2} G(\tau_1-\tau_2)\frac{\delta}{\delta h(\tau_1)}\frac{\delta}{\delta \mathrm{i}\hat{\sigma}(\tau_2)}\right)\exp\left(\int_\tau \mathrm{i}\hat{\sigma}(\tau)m(\tau)\right)\right]\Bigg|_{h(\tau)=b} \tag{11}$$

where $m(t)$ is given by equation (5) and the averaged spin autocorrelation function must be determined self-consistently from equation (6). The case $\lambda = 0$ can be solved exactly. This yields for the response function

$$G(t-t') = \theta(t-t')\Gamma\, \mathrm{e}^{\Gamma(t-t')} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\, \beta \cosh^{-2}[\beta(b+Jz)] \tag{12}$$

which is the same expression as found by Kree and Zippelius [7] for a model of asymmetric diluted neural networks in the limit of strong dilution. For temperature $T=0$ ($\beta \to \infty$) and external field $b=0$ the spin autocorrelation function $C(t)$ is given in the following implicit form (time t as a function of C):

$$t=\Gamma^{-1}\int_{C(t)}^{1} \mathrm{d}x\left(x^2+\frac{4}{\pi}[1-x\sin^{-1}x-(1-x^2)^{1/2}]\right)^{-1/2}. \tag{13}$$

A numerical integration yields a minimal deviation of $C(t)$ from an exponential decay:

$$C(t)\approx \exp\{-\Gamma[(\pi-2)/\pi]t\}. \tag{14}$$

This seems to contradict the result of Bausch *et al* [8], who find for the same model but within the Langevin approach a non-equilibrium freezing transition with non-vanishing EA order parameter $q\equiv \lim_{t\to\infty} C(t)$ in the absence of thermal noise. However, their result is due to the fact that in their treatment not only the temperature goes to zero, but also the spin flip rate Γ. In this trivial case there is no dynamics at all; equation (11) also predicts $C(t)=C(t=0)=1$.

If we assume the external local fields b_i—corresponding to the threshold value of the ith neuron—to be distributed independently according to a Gaussian distribution with zero mean and variance B, which is more realistic for brain models than a vanishing external field, then we get a non-vanishing EA order parameter. With $\mu \equiv B^2/J^2$, we get $q(\mu)=[2/(\pi-2)]\mu+O(\mu^2)$ for $\mu \ll 1$ and $q\to 1$ for $\mu\to\infty$. The relaxation time for the autocorrelation function becomes smaller for $\mu\neq 0$ and Γ^{-1} for $\mu\to\infty$.

We have studied the effect of weak correlations ($|\lambda|\ll 1$) between the synaptic couplings (J_{ij}, J_{ji}) by expanding the response function G and the autocorrelation function C around the saddle point C_0, G_0 at $\lambda=0$ and neglecting terms of higher order than first in λ. Within this approximation we find for the effective relaxation time of the response function:

$$\Gamma_{\mathrm{eff}}^{-1}\equiv \mathrm{i}\frac{\partial}{\partial\omega}\left(\frac{G(\omega)}{G(\omega=0)}\right)^{-1}\Bigg|_{\omega=0}=\Gamma^{-1}\frac{1+\lambda(2/\pi)(1-q^2)^{-1/2}}{1-\lambda(2/\pi)(1-q^2)^{-1/2}}. \tag{15}$$

If $\lambda\leqslant 0$ the relaxation time is always finite, whereas for $\lambda>0$ it is finite only as long as q is small enough. So we have investigated the long-time limit q of the autocorrelation function for $\lambda\neq 0$. We find a lower limit for λ below which the solution $q=0$ is stable. This limit is $\lambda_c:=\frac{1}{8}\pi(\pi-2)\approx 0.448$. For $\lambda\geqslant\lambda_c$ the EA order parameter as a function of λ is continuous, but the derivative has a jump at λ_c from zero to a non-vanishing value.

Summarising our results on the asymmetric SK model we have found for the fully asymmetric SK model that there exists no spin glass phase with diverging relaxation times or non-vanishing EA order parameter in zero field, not even in the deterministic case ($T=0$). Hertz *et al* [5] suggested the non-existence of diverging relaxation times in the asymmetric SK model, but their argument is not rigorous, as they point out, except in the limit $m\to\infty$ (where m is the number of spin components). Within first-order perturbation theory in the correlation parameter λ we were able to determine at $T=0$ the relaxation time of the response function and the EA order parameter, which vanishes for small λ. The relaxation time is finite for λ small enough, which means that the asymmetry in the synaptic couplings is strong enough. Although we were not able to calculate the relaxation time of the autocorrelation function, it seems to us that there is no spin glass transition in the asymmetric SK model, at least with a strong asymmetry.

After completion of this work, we became aware of the recent work of Crisanti and Sompolinsky [9], who studied the Langevin dynamics of asymmetric networks and found the absence of a spin glass phase in general networks with Gaussian asymmetry.

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